

# Final Exam Sample Questions, MA3042 Fall 2004

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1. Let  $A = \begin{bmatrix} \alpha & 1 & i \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ . For precisely what values of  $\alpha$  is  $A$  diagonalizable? Explain.

2. Let  $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 10 \end{bmatrix}$ .

Use the Cholesky factorization  $A = LL^T$  to solve  $A\mathbf{x} = (4, -8, 13)^T$ .

3. Let  $f(x, y, z) = x^3 + xyz + y^2 - 3x$ .

- Find the Hessian of  $f$  at  $\mathbf{x}_0 = (1, 0, 0)^T$  and, along the way, verify that  $(1, 0, 0)$  is a stationary point for  $f$ .
- Is the matrix from (a) positive definite? Explain.
- Does  $f$  have a maximum, a minimum, or a saddle point at  $(1, 0, 0)$ ?

4. Let  $A = \begin{bmatrix} 2 & 2 & 4 & -2 & 5 & 2 \\ 0 & 4 & -2 & 6 & 10 & 0 \\ 0 & 0 & 1 & 5 & 9 & -7 \\ 0 & 0 & 0 & 2 & 3 & 9 \\ 0 & 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 2 & 8 & -4 \end{bmatrix}$ .

Construct a matrix  $H$  with the property that the fifth and sixth entries in column four of  $HA$  are 0's.

5. Let  $S = \text{Span}((1, 3, -2)^T, (0, 1, 4)^T)$ .

- Find a basis for  $S^\perp$ .
- Let  $\mathbf{x} = (27, -10, 8)^T$ . Express  $\mathbf{x}$  as a sum  $\mathbf{x} = \mathbf{y} + \mathbf{z}$ , where  $\mathbf{y} \in S$  and  $\mathbf{z} \in S^\perp$ . Is your solution unique? Explain.

6. Suppose  $A$  and  $B$  are  $n \times n$  matrices, and additionally suppose that  $A$  is skew-Hermitian and  $B$  is real and symmetric.

- Show that  $A + B$  is not necessarily normal.
- Does your conclusion change if  $A$  and  $B$  commute, i.e., if  $AB = BA$ ?

7. Suppose that we have experimental data as shown in the following table.

$x$	-1	0	1	2
$y$	0	0	1	1

Give the equation of the line that most nearly (in the least-squares sense) passes through all four data points.

8. Let  $f(x, y, z) = 2x^2 + 8xy - 4xz + 11y^2 + 4yz + 54z^2$ .

- Find the matrix  $A$  corresponding to  $f$ .
- Determine whether  $A$  is positive definite by either finding the Cholesky factorization of  $A$  or finding that none exists.

- (c) Does  $f$  have a global maximum, a global minimum, or a saddle point at the origin?
9. Let  $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 2 & 4 \end{bmatrix}$ .
- Find the singular value decomposition of  $A$ .
  - Produce orthonormal bases for  $R(A)$ ,  $R(A^T)$ ,  $N(A)$ , and  $N(A^T)$ .
  - Construct the best rank-one approximation of  $A$ .
10. Recall that a square matrix  $A$  is *skew-Hermitian* if  $A^H = -A$ . Similarly,  $A$  is *skew-symmetric* if  $A^T = -A$ . Show that  $A \in \mathbf{C}^{n \times n}$  is skew-Hermitian if and only if  $A = B + iC$ , where  $B$  is real and skew-symmetric and  $C$  is real and symmetric.
11. Let  $A = \begin{bmatrix} 2 & 3 & 5 \\ -2 & -3 & 5 \\ -2 & -2 & 5 \end{bmatrix}$ , and  $\mathbf{b} = \begin{bmatrix} 6 \\ 14 \\ 12 \end{bmatrix}$ . Construct the decomposition  $PA = LU$ , using partial pivoting, and use this decomposition to solve  $A\mathbf{x} = \mathbf{b}$ .
12. Let  $f(x, y) = 2x^3 + x^2 + 2y^2 - 4xy + 2$ .
- Verify that  $(0, 0)$  and  $(1/3, 1/3)$  are stationary points for  $f$ .
  - Construct the Hessian of  $f$  for each stationary point.
  - Does  $f$  have a maximum, a minimum, or a saddle point at  $(0, 0)$ ? At  $(1/3, 1/3)$ ?
13. Let  $S$  be a subspace of  $\mathbf{R}^n$ , and suppose that  $P$  is the projection matrix that projects each vector  $\mathbf{x} \in \mathbf{R}^n$  onto its orthogonal projection  $P\mathbf{x} \in S$ . Let  $Q$  be the matrix that projects each  $\mathbf{x} \in \mathbf{R}^n$  into  $S^\perp$ . Show that  $Q = I - P$ .
14. Suppose that  $A$  is a real  $5 \times 4$  matrix with singular value decomposition given by

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \mathbf{v}_3^T \\ \mathbf{v}_4^T \end{bmatrix}.$$

Assume that  $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$ .

- What is the rank of  $A$ ?
  - How is  $\mathbf{v}_1$  computed?
  - How is  $\mathbf{u}_1$  computed?
  - Show that  $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \sigma_3 \mathbf{u}_3 \mathbf{v}_3^T$ , and that we can therefore obtain a more compact SVD for  $A$  by discarding columns 4 and 5 of  $U$ , rows 4 and 5 and column 4 of  $\Sigma$ , and column 4 of  $V$ .
  - What matrix  $B$  is the best rank-one approximation to  $A$ ?
15. Let  $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$ . In the singular value decomposition of  $A$ , we have  $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$ , and can use
- $$V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

- (a) Find  $U$  such that  $A = U\Sigma V^T$ .
  - (b) From the columns of  $U$  and  $V$ , produce orthonormal bases for  $R(A)$ ,  $R(A^T)$ ,  $N(A)$ , and  $N(A^T)$ .
  - (c) Use the singular value decomposition of  $A$  to construct the best (in terms of  $\|\cdot\|_F$ ) rank-one approximation of  $A$ .
  - (d) Based on (c), write  $A$  as a sum  $A = \sigma_1 A_1 + \sigma_2 A_2$ , where  $A_1$  and  $A_2$  are rank 1 matrices.
16. Let  $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$  be an ordered basis for  $\mathbf{R}^3$ . Define  $L : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  by

$$L(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2 + \gamma \mathbf{u}_3) = (\alpha - 2\gamma)\mathbf{u}_1 + (2\alpha + \beta)\mathbf{u}_2 + (\alpha + 4\beta - \gamma)\mathbf{u}_3.$$

Find the matrix  $A$  representing  $L$  with respect to  $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ .

17. Let  $L$  be the operator on  $P^3$  defined by  $L(p(x)) = p'(x) + x^2 p''(x)$ .
- (a) Describe the kernel and the range of  $L$ .
  - (b) Construct the matrix representation  $A$  of  $L$  with respect to the basis  $[1, x, x^2]$ .
  - (c) Construct the matrix representation  $B$  of  $L$  with respect to the basis  $[1, x, x + x^2]$ .
  - (d) Find the matrix  $S$  such that  $B = S^{-1}AS$ . (To verify, show that  $SB = AS$ .)

18. Let  $A = \begin{bmatrix} 1 & -1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 3 & 1 \\ 5 & -2 & 4 & 1 & -1 \\ 7 & -1 & 2 & -1 & 0 \\ 3 & 0 & 2 & 1 & 1 \end{bmatrix}$ .

Show that 7 is an eigenvalue of  $A$ , by finding an associated eigenvector.